

BROAD POSETS, TREES, AND THE DENDROIDAL CATEGORY

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ABSTRACT. An extension of order theory is presented that serves as a formalism for the study of dendroidal sets analogously to way the formalism of order theory is used in the study of simplicial sets.

1. INTRODUCTION

In algebraic topology a space X is often replaced by its singular complex $S(X)$ which is defined as follows. For every $n \geq 0$ let Δ^n be the standard n -dimensional simplex. The singular complex $S(X)$ has, at each dimension $n \geq 0$, the set $S(X)_n = \{f : \Delta^n \rightarrow X \mid f \text{ is continuous}\}$. A famous result due to Quillen shows that for homotopy theoretic purposes it makes little difference whether one works directly with the space X or its singular complex $S(X)$. The advantage of working with $S(X)$ is that it is a completely combinatorial object belonging to the theory of simplicial sets. The combinatorics within a simplicial set is governed by the polytopal interrelations between the standard n -simplices $\{\Delta^n \mid n \geq 0\}$. As is well known, these interrelations are equivalent to those between all finite, non-empty, linearly ordered sets. The latter observation brings to the theory of simplicial sets, and thus to topology, the very rigorous and algebraic formalism of order theory.

Recently ([9]), the concept of simplicial set was generalized to that of dendroidal set. The context of the generalization is operadic rather than topological but the general aim is the same: to provide for combinatorial models of operads and ∞ -operads. The combinatorics within a dendroidal set is governed by the interrelations between finite trees. The theory of dendroidal sets is then seen to extend that of simplicial sets by viewing every finite linear order as a linear tree.

The aim of this work is to present an extension of order theory that stands in the same relation to dendroidal sets as order theory does to simplicial sets. We exemplify how the extension we present can be used to argue about trees, and thus dendroidal sets, in a way that is analogous to the use of posets in arguments about simplicial sets. We thus provide a rigorous and algebraic formalism playing the same role in the theory of dendroidal sets as order theory does in the theory of simplicial sets.

Though we develop just that part of the theory of the extension of order theory that is needed for its applicability to dendroidal sets, we note that the theory appears to be interesting in its own right.

Plan of the paper. Section 2 contains an intuitive description of trees and operads stressing the aspects that are relevant to the formalism presented. Section 3 then contains the extension of order theory which is the ambient category where the main objects of study, dendroidally ordered sets, presented in Section 4, reside. Section 5 then classifies dendroidally ordered sets in terms of trees, with Section 6

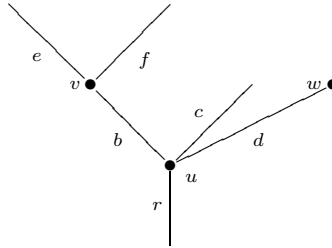
closing the article with a proof, completely within the formalism developed below, of a fundamental decomposition result in the dendroidal category.

2. TREES AND OPERADS

Trees and operads are presented in a rather intuitive fashion meant to immediately make available the key ideas relevant to the following sections. For more detailed accounts of operads given in the spirit of dendroidal sets the reader is referred to [8] or [11].

2.1. Trees. By a tree we mean a rooted, either planar or non-planar, tree with leaves and stumps. Various definitions of tree exist in the literature and perhaps the two most common approaches are to define a tree as a special graph or as a special topological space. However, even within a single framework there are many possibilities for a formalization of the tree concept. For instance, it is common to define a tree as consisting of a set E of edges and a set V of vertices together with certain incidence relations and some conditions. But, it is also possible (see [11]) to dispose of the set V and capture vertices as a by-product of a certain structure only on the set of edges. Expectedly, different formalisms have virtues in different situations.

The picture



of a tree exemplifies all of the features of interest to us. It consists of six *edges* of which the one labelled r is the *root*. It has three *vertices* marked by \bullet and each vertex has a set of *incoming* edges and single *outgoing* edge leading to the root. One of these vertices, labelled w , contains no incoming edges and is called a *stump*. The edges labelled c, e, f have no vertex at their top and are called *leaves*. The picture can be taken as that of either a *planar* tree or a *non-planar* one, with the crucial difference being whether or not the order of the incoming edges at each vertex is important or not, namely, in the planar case it is important and in the non-planar one it is not. Certain parts of the tree lie at its outermost layer, such as the pair of edges $\{e, f\}$ as well as the stump w . Such regions of the tree are called *external clusters*. Intuitively, these are parts of the tree that can be trimmed by removing a single vertex completely. Edges that belong to the inner layers of the tree are called *inner edges*. More precisely, these are edges with a vertex at each end, such as edges b and d . Intuitively, an inner edge is an edge that can be contracted to merge together two vertices. Other aspects visible in the tree above are that every edge that is not a leaf has *children* and that every edge other than the root is a child of a unique *parent* edge. Every two edges admit a *join*, an edge which is the first *common ancestor* of the given edges. For instance, the join of e and f is b while that of e and c is r . Lastly, an intuitive feature of trees is that two trees can

be *grafted* by placing one on top of the other and identifying the root in one with a leaf in the other.

2.2. Operads. By an operad we mean either a symmetric or planar coloured operad, also known as a multicategory. Intuitively, it consists of a class of labelled trees (planar ones for planar operads and non-planar ones for symmetric operads) such that the edges in a tree are labelled by objects and the vertices are labelled by multivariable arrows. The various labelled trees must satisfy a consistency condition that basically says that each multivariable arrow has a unique input and output. Moreover, the class of labelled trees must be saturated, meaning that every finite combinations of multivariable arrows with matching inputs and outputs occur as a labelled tree. On top of that structure there is then a composition operation that turns one such labelled tree into a labelled tree with just one vertex and having the same number of leaves as the original tree. For the composition there is an associativity condition that says that starting with a single labelled tree, composing it in one go or composing any subtrees of it first will result in the same composition (and there are also identity constraints which we neglect in this intuitive explanation). As such, any tree naturally gives rise to an operad by generating a free one. The objects are then the edges of the tree and the arrows are freely generated by the vertices in the tree. Stumps are then interpreted as constants.

We mention a few trees that play an important role in the theory. A tree L_n of the form

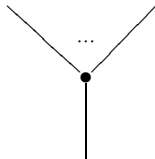


with one leaf and only unary vertices is a *linear tree*. The linear trees stand to all trees in the same way that ordinary functions stand to multivariable functions. The special case of the tree L_0



consisting of just one edge and no vertices is denoted by η and is the only tree, up to isomorphism, whose root is also a leaf.

A tree C_n of the form



that has just one vertex and n leaves is called an *n-corolla*. The corollas can be seen as the building blocks of all trees as any tree is the grafting of corollas. A

related remark is that any tree T admits an essentially unique decomposition as the grafting $C_n \circ (T_1, \dots, T_n)$ where each T_i is the subtree of T having as its root the i -th incoming edge to the root of T .

In the study of operads, and particularly ∞ -operads, the dendroidal category plays a prominent role (see [9, 10, 8, 11, 4, 2, 3]). It comes in two flavours depending on whether one studies planar or symmetric operads. The planar dendroidal category Ω_π has as objects all planar trees and as arrows all maps of operads between the operads generated by the trees. Similarly, the non-planar dendroidal category Ω has non-planar trees as objects and symmetric operad maps as arrows. The interrelations between the trees in (each variant of) the dendroidal category is what we call the operadic tree combinatorics.

To illustrate what we achieve below, consider the following. A common definition of tree that uses the language of order theory is as a poset P that satisfies that for every $x \in P$ the down set $x_\downarrow = \{y \in P \mid y \leq x\}$ is well-ordered. This definition does not capture the operadic tree combinatorics. For instance, from the operadic point of view, an n -corolla C_n has precisely $n + 1$ subtrees, all having just a single edge (which represent the inputs and output of a multivariable arrow). However, using the definition just mentioned, C_n would also have n linear subtrees with two edges (which do not correspond to anything one can obtain from a multivariable arrow).

Another formalism of trees which exhibits the same kind of difficulty is that given in [5] where a tree is defined as a topological space. Recently, a formalism of trees in terms of polynomial functors was given in [7] which does capture the operadic tree combinatorics.

The aim of this work, which expands on ideas introduced by the author briefly in [11], is to develop an order theoretic formalism in which all of the intuitive properties of trees above follow from just three axioms and such that the order preserving functions capture the combinatorics of trees relevant to operads. Such a formalism allows for very precise arguments about dendroidal sets that do not rely on sometimes vague intuition about trees and can be useful in other places where tree formalisms are needed.

3. BROAD POSETS

This section presents the notion of broad poset, exhibits ordinary posets as a slice of broad posets, relates the latter to operads, and establishes a closed symmetric monoidal structure by means of a suitable tensor product of broad posets.

3.1. Definition of broad posets and their relation to operads. For a set A we denote by A^\cdot the free monoid on A with unit ϵ . The free commutative monoid A^+ is obtained from A^\cdot by an obvious abelianization process. We use the same notation $a \cdot b$ to indicate both the operation in A^\cdot as well as in A^+ . We identify A in either A^\cdot or A^+ in the obvious way.

Definition 3.1. A *commutative broad relation* is a pair (A, R) where A is a set and R is a subset of $A^+ \times A$. A *non-commutative broad relation* is a pair (A, R) where A is a set and R is a subset of $A^\cdot \times A$.

All of the definitions and results below come in a commutative as well as a non-commutative flavour, with the formulation essentially unchanged. Thus, we use the notation A^* to mean that it can be replaced, throughout an entire definition

or result, by either A^- or A^+ . The same convention holds for the use of the term 'broad relation'. It can be replaced throughout a definition or result by either 'non-commutative broad relation' or 'commutative broad relation'. The following is an instance of this convention. As is common with ordinary relations, for $a_1 \in A^*$ and $a_2 \in A$, we write $a_1 R a_2$ to mean $(a_1, a_2) \in R$, for any broad relation R . We also write $a_2 \in a_1$ to indicate that a_2 occurs in a_1 (as a factor in the non-commutative case and as a summand in the commutative case).

Definition 3.2. A *broad poset* is a broad relation (A, R) such that, for all $n \geq 0$, $a_1, \dots, a_n, a, a' \in A$ and $b_1, \dots, b_n \in A^*$, the following conditions hold.

- Reflexivity: $a R a$.
- Transitivity: If $a_1 \cdot \dots \cdot a_n R a$ and, for all $1 \leq i \leq n$, $b_i R a_i$ hold then $b_1 \cdot \dots \cdot b_n R a$ holds.
- Anti-symmetry: If $a R a'$ and $a' R a$ both hold then $a = a'$.

Remark 3.3. Following our convention, this definition is actually two definitions. When A^* is the free monoid on A then the notion defined is called a *non-commutative broad poset*. When A^* is the free commutative monoid on A then the notion defined is called a *commutative broad poset*. Once more, the term 'broad poset' can be replaced throughout by either 'commutative broad poset' or 'non-commutative broad poset'.

When (A, R) is a broad poset we denote R by \leq and then the meaning of $<$ is defined in the obvious way. Obviously, one has the standard constructions, for a broad relation R , of the *reflexive closure* $R^r = R \cup \{(a, a) \mid a \in A\}$ and the *transitive closure* $R^t = \bigcap_{R \subseteq_t S} S$ (where the notation $R \subseteq_t S$ indicates that S is a transitive broad relation containing R). Lastly, if R satisfies reflexivity and transitivity then setting $a \sim b$, for $a, b \in A$, when both $a R b$ and $b R a$ hold, defines an equivalence relation and $R_0 = R / \sim$ inherits the broad relation structure from R . It follows easily that if R is any broad relation then $(R^t)_0$ is a broad poset, called the broad poset *generated* by R .

Definition 3.4. A function $f : A \rightarrow A'$ between broad posets is *monotone* if, for every $b \in A^*$ and $a \in A$, the inequality $b \leq a$ implies $f(b) \leq f(a)$ (where $f(b) = f(b_1 \cdot \dots \cdot b_n) = f(b_1) \cdot \dots \cdot f(b_n)$).

We thus obtain the categories \mathbf{bPos}_c and \mathbf{bPos}_π of, respectively, commutative and non-commutative broad posets and monotone functions. In accordance with our convention, the term \mathbf{bPos} below is meant to be replaced throughout by either \mathbf{bPos}_c or \mathbf{bPos}_π .

Remark 3.5. Recall that preordered sets and monotone functions are equivalent to categories enriched in the truth values monoidal category $V = \{F < T\}$. Similarly, commutative broad preorders, i.e., commutative broad relations satisfying reflexivity and transitivity, are essentially the same as symmetric operads enriched in V . In the same vein, non-commutative broad preorders, i.e., non-commutative broad relations satisfying reflexivity and transitivity, are essentially the same as planar operads enriched in V .

Example 3.6. There is a whole range of possible broad poset structures on a singleton set, of which we mention two. A terminal object, $*$, in \mathbf{bPos} is a singleton set $S = \{*\}$ in which, if we write $n \cdot * = * \cdot \dots \cdot *$ for the n -fold product/sum, the

inequality $n \cdot * \leq *$ holds for every $n \geq 0$. At the other extreme we find the broad poset $*$ in which $n \cdot * \leq *$ holds if, and only if, $n = 1$. Notice that there is, for every broad poset A , a bijection between the set of monotone functions $*$ \rightarrow A and the elements of A .

Example 3.7. If P is a meet semi-lattice then one can define a broad poset structure on P as follows. Given $p_0, \dots, p_n \in P$, the inequality $p_1, \dots, p_n \leq p_0$ holds precisely when $p_1 \wedge \dots \wedge p_n \leq p_0$. More generally, if (P, \cdot, I) is a symmetric monoid object in **Pos** then one similarly obtains a commutative broad poset structure on P . We mention that broad posets arising from symmetric monoid objects in **Pos** can be characterized by certain representability conditions in a way similar to the representability of symmetric multicategories given in [6]. Similar remarks are valid in the non-commutative case.

Example 3.8. For every $n \geq 0$ let γ_n be a set $\{r, l_1, \dots, l_n\}$ with $n+1$ elements with the broad poset structure in which the only inequality, other than those imposed by reflexivity, is $l_1 \cdot \dots \cdot l_n \leq r$. Note that, for every broad poset A , a monotone function $\gamma_n \rightarrow A$ corresponds bijectively to a choice of $n+1$ elements $a_0, \dots, a_n \in A$ satisfying $a_1 \cdot \dots \cdot a_n \leq a_0$. The broad poset γ_n is called an *n-corolla*.

Theorem 3.9. *The category **bPos** is small complete and small cocomplete.*

Proof. One may easily construct all required limits and colimits directly. A more conceptual argument uses Remark 3.5 above. Since the truth values category $V = \{F < T\}$ is complete and cocomplete it follows from general considerations of enriched operad theory that the category **bPreOrd** of broad preorders is small complete and small cocomplete. This suffices to construct all small limits in **bPos**. To obtain small colimits in **bPos** one needs to also employ the functor $(-)_0 : \mathbf{bPreOrd} \rightarrow \mathbf{bPos}$, obtained by the construction $R \mapsto R_0$ described above. \square

Remark 3.10. Below we show that the cartesian structure on **bPos** is not closed. We thus describe explicitly the product of two broad posets A, B . Such a product is obtained by endowing the set $A \times B$ with the broad relation where $(a_1, b_1) \cdot \dots \cdot (a_n, b_n) \leq (a, b)$ holds precisely when the two inequalities $a_1 \cdot \dots \cdot a_n \leq a$ and $b_1 \cdot \dots \cdot b_n \leq b$ hold. The terminal object $*$ was discussed in Example 3.6.

Note that the category **Pos** can be recovered, up to equivalence, from **bPos** by slicing over the broad poset $*$ (this is simply the trivial observation that the unique function $A \rightarrow *$ is monotone if, and only if, A is essentially an ordinary poset). The forgetful functor $\mathbf{bPos}/\star \rightarrow \mathbf{bPos}$ gives an embedding $k_! : \mathbf{Pos} \rightarrow \mathbf{bPos}$ which is easily seen to have a right adjoint $k^* : \mathbf{bPos} \rightarrow \mathbf{Pos}$. This right adjoint k^* sends a broad poset (A, R) to the poset (A, S) where for $a, a' \in A$ holds aSa' precisely when aRa' holds.

Obviously, there is a forgetful functor $\Sigma^* : \mathbf{bPos}_c \rightarrow \mathbf{bPos}_\pi$ (induced by the evident surjection $A^- \rightarrow A^+$) whose left adjoint $\Sigma_! : \mathbf{bPos}_\pi \rightarrow \mathbf{bPos}_c$ sends a non-commutative broad poset R to its abelianization.

Recall that a poset A can be considered as a category \mathcal{C} whose objects are the elements of A and such that there is precisely one arrow $a \rightarrow a'$ in \mathcal{C} whenever $a \leq a'$. One obtains thus a functor $\mathbf{Pos} \rightarrow \mathbf{Cat}$. Similarly, given a broad poset B

one can define a (symmetric or planar) operad \mathcal{P} whose objects are the elements of B and such that there is exactly one operation in $\mathcal{P}(b_1, \dots, b_n; b)$ whenever $b_1 \cdots b_n \leq b$. In that way one obtains the functors $\mathbf{bPos}_c \rightarrow \mathbf{Ope}$ and $\mathbf{bPos}_\pi \rightarrow \mathbf{Ope}_\pi$. We summarize the properties of these constructions in the following theorem.

Theorem 3.11. *In the diagram*

$$\begin{array}{ccccc}
 \mathbf{bPos}_c & \xrightarrow{\quad} & \mathbf{Ope} \\
 \uparrow \Sigma_! & \swarrow k_! & \nearrow j_! & \uparrow \Sigma_! \\
 & \mathbf{Pos} & \longrightarrow & \mathbf{Cat} & \\
 \downarrow \Sigma^* & \nwarrow k^* & \nearrow j^* & \downarrow \Sigma^* \\
 \mathbf{bPos}_\pi & \xrightarrow{\quad} & \mathbf{Ope}_\pi
 \end{array}$$

all pairs of arrows are adjunctions (with left adjoint on the left or on top) and each of the four triangles that consist of just left or just right adjoints commutes. The horizontal arrows are embeddings and each of the two trapezoids commutes. Moreover, each of the right adjoints other than the left most vertical one is equivalent to the canonical forgetful functor of a slice category.

Proof. We omit the proofs of the claims not given above and refer the reader to [11] for more information on some of the properties concerning the triangles on the right. \square

3.2. Tensor products. The category \mathbf{Pos} is cartesian closed with the straightforward definition of products of posets. The internal hom, for two posets $P, Q \in \mathbf{ob}(\mathbf{Pos})$, is the poset $[P, Q]$ of all monotone functions $f : P \rightarrow Q$ where $f \leq g$ holds precisely when, for all $p \in P$, the inequality $f(p) \leq g(p)$ holds. This monoidal structure is inherited from the closed cartesian structure on \mathbf{Cat} along the embedding $k_! : \mathbf{Pos} \rightarrow \mathbf{Cat}$. It is known that the category \mathbf{Ope} is cartesian but not cartesian closed and that it does possess a symmetric closed monoidal structure, given by the Boardman-Vogt tensor products ([1]), that restricts along $j_! : \mathbf{Cat} \rightarrow \mathbf{Ope}$, to the cartesian product of categories. We now show that similar results are true for broad posets.

Proposition 3.12. *The category \mathbf{bPos} is cartesian but not cartesian closed.*

Proof. By Theorem 3.9 the category \mathbf{bPos} has all small products and is thus cartesian. To show that the cartesian structure (given explicitly in Remark 3.10) is not closed recall the definition of corollas from Example 3.8 and consider the pushout

$$\begin{array}{ccc}
 \star & \longrightarrow & \gamma_2 \\
 \downarrow & & \downarrow \\
 \gamma_2 & \longrightarrow & X
 \end{array}$$

where one of the arrows $\star \rightarrow \gamma_2$ chooses $r \in \gamma_2$ and the other one chooses $l_1 \in \gamma_2$. It is easy to see that this pushout is not preserved under the functor $\gamma_3 \times - : \mathbf{bPos} \rightarrow \mathbf{bPos}$, thus proving the claim. \square

Definition 3.13. Let A and B be two broad posets. Their *tensor product* $A \otimes B$ is the set $A \times B$ with the broad poset generated by the broad relation in which

- for every $a \in A$ if $b_1 \cdots b_n \leq b$ then $(a, b_1) \cdots (a, b_n) \leq (a, b)$, and
- for every $b \in B$ if $a_1 \cdots a_m \leq a$ then $(a_1, b) \cdots (a_m, b) \leq (a, b)$.

Note that these defining relations guarantee that for every $a \in A$ the function $a \otimes - : B \rightarrow A \otimes B$, given by $b \mapsto (a, b)$, is monotone and similarly that for every $b \in B$ the function $- \otimes b : A \rightarrow A \otimes B$, given by $a \mapsto (a, b)$, is monotone.

Theorem 3.14. *The category \mathbf{bPos} with the tensor product of broad posets is a symmetric closed monoidal category, and $k_! : \mathbf{Pos} \rightarrow \mathbf{bPos}$ is strong monoidal.*

Proof. The broad poset \star is clearly a unit for the tensor product and it is easily verified that \otimes makes \mathbf{bPos} into a symmetric monoidal category, so all that is left to do is describe the internal hom. Given two broad posets A and B , the set $[A, B]$ of all monotone functions $f : A \rightarrow B$ is made into a broad poset by setting $f_1 \cdots f_n \leq f$ precisely when for every $a \in A$ the inequality $f_1(a) \cdots f_n(a) \leq f(a)$ holds. It is routine to verify that this broad poset is the required internal hom. The fact that $k_! : \mathbf{Pos} \rightarrow \mathbf{bPos}$ is strong monoidal is trivial. \square

Returning to the diagram of Theorem 3.11, we see that all of the categories there are equipped with symmetric closed monoidal structures given by the Boardman-Vogt tensor product of operads and tensor product of broad posets (for the corner categories), and the cartesian structure (for the remaining two). With these monoidal structures, each functor labeled by a $-_!$ is strong monoidal. The precise monoidal behaviour of the other functors is omitted here except for the following interesting observation. Given commutative broad posets A, B the formula

$$\Sigma_!(\Sigma^* A \otimes \Sigma^* B) \cong A \otimes B$$

holds. The same formula does not remain valid if A and B are symmetric operads. This phenomenon is related to the fact that, in the above diagram, the triangle on the left is not quite a slice of the triangle on the right. Operads have a much greater expressive power than broad posets do at a cost of requiring more elaborate structure. That extra structure, in those operads that are essentially broad posets, manifests itself by redundancy (e.g., the functor $\mathbf{bPos}_c \rightarrow \mathbf{Ope}$ sends \star not to the terminal operad \mathbf{Comm} but rather to the operad \mathbf{As} having just one object but $n!$ arrows of arity n for each $n \geq 0$). The lack of this redundancy in broad posets allows for the formula above.

4. DENDROIDALLY ORDERED SETS

This section introduces the main concept of this work, that of a dendroidally ordered set, as a broad poset satisfying three axioms. Several of the intuitive tree notions from Section 2 are established as consequences of these axioms to be used in the subsequent sections, and a definition of the dendroidal category is given in terms of dendroidally ordered sets and monotone functions.

4.1. The tree formalism. A broad poset \leq induces a partial order relation on the set A^* as follows. For $a, b \in A^*$ we say that $a \leq b$ if $b = b_1 \cdots b_n$, with each $b_i \in A$, and if there exist $a_1, \dots, a_n \in A^*$ such that $a = a_1 \cdots a_n$ and such that $a_i \leq b_i$ holds for each $1 \leq i \leq n$. Notice that it is harmless to use the same symbol \leq both for the broad poset on A and for the induced relation on A^* .

For $a, b \in A$ we say that b is a *descendent* of a and write $b \leq_d a$, if there is some $b' \in A^*$ such that both $b' \leq a$ and $b \in b'$ hold. Clearly, \leq_d is a preorder on A . If it

is a poset then we say that the broad poset \leq is *stratified*. A broad poset (A, \leq) is *finite* if the set \leq is finite, in which case it is automatically stratified. For an element $a \in A$ we write $\hat{a} = \{b \in A^* \mid b < a\}$.

Definition 4.1. Let A be a broad poset and $a \in A$. If $\hat{a} = \emptyset$ then a is called a *leaf*. Otherwise, if \hat{a} has a maximum, denoted by a^\uparrow , then a is said to *have children* and each element in a^\uparrow is a *child* of a .

Clearly it is not always the case that an element $a \in A$ is either a leaf or has children.

Remark 4.2. Notice that it is possible that $a^\uparrow = x \in A$. More importantly, it is also possible that $a^\uparrow = \epsilon$, the monoid unit. In that case, a is not a leaf nor does it have any element $x \in A$ as a child. Such an a is called a *stump*, the existence of which is an important aspect of the formalism that agrees with the interpretation, in operad theory, of 0-ary operations as constants. In each of these cases it is grammatically incorrect to say that a has children but we will ignore such linguistic difficulties.

Definition 4.3. A *dendroidally ordered set* is a finite broad poset A satisfying, for all $a_1, \dots, a_n, a \in A, n \geq 0$, the following three conditions.

- \leq is *simple* in the sense that if $a_1 \cdots a_n \leq a$ then $a_i = a_j$ implies $i = j$.
- The poset (A, \leq_d) has a minimal element r_A called the *root*.
- If a is not a leaf then it has children.

Conforming with our convention this definition actually defines two concepts: commutative and non-commutative dendroidally ordered sets. The use of the term 'dendroidally ordered set' is meant to be replaced throughout by one of the two.

When A is dendroidally ordered we will also refer to its elements as *edges*. The following useful proposition establishes several of the intuitive concepts of trees described Section 2 as consequences of the axioms.

Proposition 4.4. *For every dendroidally ordered set A and edges $a, a_1, a_2, b \in A$ the following properties hold.*

- (1) If $a <_d b$ then there is a unique child $t \in b^\uparrow$ for which $a \leq_d t$.
- (2) If a_1, a_2 are descentance incomparable then the inequalities $a_1 \leq_d b$ and $a_2 \leq_d b$ together imply the existence of a single $c \in A^*$ for which both $a_1, a_2 \in c$ and $c \leq b$ hold.
- (3) If a is not the root then $a \in x^\uparrow$ holds for a unique edge $x \in A$, called its *parent*.
- (4) The poset (A, \leq_d) has all binary joins.

Proof.

□

- (1) $a <_d b$ implies that for some $x \in A^*$ both $a \in x$ and $x \leq b$, and so $x \in \hat{b}$. Thus, $x \leq b^\uparrow$ which, by definition, implies that $a \leq_d t$ for some $t \in b^\uparrow$. Uniqueness follows since the existence of two distinct such children contradicts simplicity.
- (2) Either $a_1 = b$ or $a_2 = b$ would imply comparability and thus there are $t_1, s_1 \in b^\uparrow$ for which both $a_1 \leq_d t_1$ and $a_2 \leq_d s_1$ hold. If $t_1 = s_1$ then repeat the argument with $b_1 = t_1$ instead of b until the first b_n where the associated t_{n+1} and s_{n+1} are distinct (which must occur since a_1 and a_2 are not comparable). Thus, we have $a_1, a_2 \leq_d t_k$ and $t_k \in t_{k-1}^\uparrow$ for all

- $0 \leq k \leq n$ (agreeing that $t_0 = b$) while $a_1 \leq_d t_{n+1}$, $a_2 \leq_d s_{n+1}$ and $t_{n+1} \neq s_{n+1}$. Using transitivity one now easily constructs the desired tuple c .
- (3) We may construct a sequence $t_1 <_d t_2 <_d t_3 < \dots$ such that $t_0 = r$ and for every $k \geq 0$ holds that $a <_d t_k$ and $t_{k+1} \in t_k^\uparrow$. Since the sequence must be finite we obtain, for the last term t_m , that $a = t_m \in t_{m-1}^\uparrow$. To prove uniqueness assume that a is a child of both x_1 and x_2 with $x_1 \neq x_2$. If $x_1 <_d x_2$ then $x_1 \leq_d t$ for some $t \in x_2^\uparrow$. One easily sees then that the case $t = a$ contradicts with \leq being finite while the case $t \neq a$ contradicts with simplicity. Similarly, $x_2 <_d x_1$ leads to a contradiction leaving us with x_1 and x_2 incomparable. But in that case find $c \in A^*$ such that $x_1, x_2 \in c$ and $c \leq r$ to obtain a contradiction by using transitivity and $a \in x_1^\uparrow$ and $a \in x_2^\uparrow$.
- (4) We may assume that a_1 and a_2 are incomparable, and thus none is the root, and proceed to construct their join. Let p_1 be the parent of a_1 and p_2 the parent of a_2 . It is not hard to see that $p_1 \vee p_2 = a_1 \vee a_2$. Thus, if p_1 and p_2 are comparable then the join is found and otherwise the same process can be repeated. This process is bounded by the root r_A and thus will terminate after a finite number of times with the desired join.

It is obvious that if $A \neq \emptyset$ is a finite linearly ordered set, then the broad poset $k_!(A)$ is dendroidally ordered. Note that \leq_d will have the empty join if, and only if, A has a single leaf, in which case the broad poset A is essentially equal to $k_!(P)$ for some linear order P .

4.2. The dendroidal category.

Definition 4.5. The *dendroidal category* Ω is the full subcategory of \mathbf{bPos} spanned by the dendroidally ordered sets.

Conforming with our convention we just defined two categories: $\Omega_c \subseteq \mathbf{bPos}_c$ and $\Omega_\pi \subseteq \mathbf{bPos}_\pi$, and Ω is intended to be replaced throughout by one of the two.

Since the simplicial category Δ is equivalent to the full subcategory of \mathbf{Pos} spanned by the finite non-empty linear orders we may use $k_!$ to identify Δ as a full subcategory of both Ω_c and Ω_π . Now consider the diagram

$$\begin{array}{ccccccc}
 \mathbf{Ope}_\pi & \xleftarrow{j_\pi} & \Omega_\pi & \xleftarrow{i} & \Delta & \xrightarrow{i} & \Omega_c \xrightarrow{j_c} \mathbf{Ope}_c \\
 & \searrow & \downarrow & & \downarrow & & \downarrow \\
 & & \mathbf{bPos}_\pi & \xleftarrow{k_!} & \mathbf{Pos} & \xrightarrow{k_!} & \mathbf{bPos}_c
 \end{array}$$

where the arrows are those discussed above (and the arrows $\Omega_c \rightarrow \mathbf{Ope}_c$ and $\Omega_\pi \rightarrow \mathbf{Ope}_\pi$ are defined to make the triangles commute). From the results below it will follow that the image of j_c is equivalent to the dendroidal category, and similarly the image of j_π is equivalent to the planar dendroidal category, defined in [8] in terms of operads.

Remark 4.6. In fact the equivalence can be strengthened to an isomorphism by considering a formalism of trees, as is done in [11], where vertices do not exist independently of the edges but rather appear as a by product of some structure on the edges.

5. TREES AND DENDROIDALLY ORDERED SETS

This section studies a grafting operation for dendroidally ordered sets that allows for precise constructions turning trees into dendroidally ordered sets and vice versa. These constructions are the object part of an equivalence of categories between the dendroidal category defined in terms of operads (as in [8]) and the one defined in terms of dendroidally ordered sets.

5.1. Grafting dendroidally ordered sets.

Definition 5.1. Let A and B be two dendroidally ordered sets, $\star \rightarrow A$ a leaf, and $\star \rightarrow B$ the root. A *grafting* of B on A , denoted by $A \circ B$, is a pushout

$$\begin{array}{ccc} \star & \xrightarrow{\quad} & A \\ \downarrow & & \downarrow \\ B & \xrightarrow{\quad} & A \circ B \end{array}$$

in **bPos**.

By renaming the elements of A and B if needed we may assume that $A \cap B = \{y\}$, where y is the chosen leaf of A and the root of B . Then a grafting is obtained as the broad poset generated by the broad relation on $A \cup B$ consisting of the relations coming from A and B . It is easily given explicitly: the inequality $z \leq x$ holds in $A \cup B$ if it holds in either A or B or if the following holds. There exists $a_1, a_2 \in A^*$ and $b \in B^*$ such that $z = a_1 b a_2$, $b \leq y$, and $a_1 y a_2 \leq x$.

It is thus easily seen that the grafting of two dendroidally ordered sets is again a dendroidally ordered set, leading to the following corollary.

Corollary 5.2. *The grafting $B \circ A$ can be computed in either **bPos** or Ω with isomorphic results.*

By repeated grafting one can define a full grafting operation

$$A \circ (B_1, \dots, B_n)$$

which is simply an n -fold pushout.

For a dendroidally ordered set A and $a \in A$ let $A_a = \{a' \in A \mid a' \leq_d a\}$ with the induced broad relation from A . It is immediate that A_a is again dendroidally ordered. For a dendroidally ordered set A with root r and $r^\uparrow = \{a_1, \dots, a_n\}$ let $A_{root} = \{r, a_1, \dots, a_n\}$, viewed as an n -corolla γ_n .

Lemma 5.3. *For a dendroidally ordered set A with root r and $r^\uparrow = \{a_1, \dots, a_n\}$ holds that $A \cong A_{root} \circ (A_{a_1}, \dots, A_{a_n})$. Moreover, this decomposition is unique in the sense that if $A \cong \gamma_m \circ (A_1, \dots, A_m)$ then $m = n$ and, up to reordering, $A_{a_i} \cong A_i$ for all $1 \leq i \leq n$.*

Proof. We show that A satisfies the universal property for the pushout $A_{root} \circ (A_{a_1}, \dots, A_{a_n})$, of which the required injections are evident. Suppose that B is any dendroidally ordered set with monotone function $A_{root} \rightarrow B$ and $A_{a_i} \rightarrow B$ making the relevant diagram commute. We need to construct an appropriate monotone function $A \rightarrow B$. By Proposition 4.4, for every $a \in A$ holds that if $a \notin A_{root}$ then $a \in A_{a_i}$ for precisely one $1 \leq i \leq n$. Moreover, the following argument shows that $A_{a_i} \cap A_{root} = \{a_i\}$. If $r \in A_{a_i}$ then it follows that $r = a_i$, but then $r \in r^\uparrow$, a contradiction (in general $a \notin a^\uparrow$ holds for every $a \in A$). If $a_j \in A_{a_i}$ and $a_j \neq a_i$

then $a_j \leq_d a_i$ which means that there is a $b \in A^*$ with $a_j \in b$ and $b \leq a_i$. But then transitivity and the inequality $r \leq (a_1, \dots, a_n)$ will contradict the simplicity of A . Thus the only element of A which can be in $A_{root} \cap A_{a_i}$ is a_i which is clearly there. These observations show that there is a unique function $A \rightarrow B$, easily seen to be monotone, which is compatible with the given monotone functions to B , completing the proof of the decomposition. The uniqueness clause follows easily. \square

Remark 5.4. Combining Corollary 5.2 and Lemma 5.3 it is seen that Ω can also be defined as the smallest full subcategory of **bPos** containing all corollas and closed under grafting.

Clearly this remark already implies that dendroidally ordered sets and trees are, in a sense, the same. To furnish an exact statement we describe constructions to turn a tree into a dendroidally ordered set and vice versa.

Let T be a tree under any formalism that allows for a precise statement of the fundamental decomposition exhibiting a tree T , essentially uniquely, as the grafting $T = T_{root} \circ (T_{e_1}, \dots, T_{e_n})$, as in Section 2. We define a dendroidally ordered set, $[T]$, whose underlying set is $E(T)$, the set of edges of T , by induction on the number k of vertices in the tree T . If $T = \eta$ (the tree with one edge and no leaves) then we set $[\eta] = \star$ while if T is an n -corolla C_n then we set $[C_n] = \gamma_n$, covering the cases $k = 0, 1$. Suppose now that T has more than 1 vertex and write $T = T_{root} \circ (T_{e_1}, \dots, T_{e_n})$. We then define $[T] = [T_{root}] \circ ([T_{e_1}], \dots, [T_{e_n}])$, where the grafting is that of dendroidally ordered sets.

For the construction associating with any dendroidally ordered set A a tree T we need the following concepts. A pair (b, a) is called a *link* in a broad poset A if $b < a$ and if for every $b' \in A^*$ the inequalities $b \leq b' < a$ imply that $b = b'$. The number of links in a broad poset A is the *degree* of A and is denoted by $d(A)$. When A is a dendroidally ordered set a link is called a *vertex*. It can easily be shown that for dendroidally ordered sets A and B the equality $d(A \circ B) = d(A) + d(B)$ holds.

To obtain a tree $Tr(A)$ from a dendroidally ordered set A we proceed by induction on $n = d(A)$. If $n = 0$ then $Tr(A) = \star$ while if $n = 1$ then $Tr(A) = \gamma_n$ where $n + 1 = |A|$, the cardinality of the set A . Assume $Tr(A)$ was constructed for all A with $d(A) < n$ and let A be a dendroidally ordered set with $d(A) = n$. Then write $A = A_{root} \circ (A_{a_1}, \dots, A_{a_n})$, and let $Tr(A) = Tr(A_{root}) \circ (Tr(A_{a_1}), \dots, Tr(A_{a_n}))$, obtained by grafting of trees.

Another straightforward inductive proof yields the following convenient degree formula, where $L(A)$ denotes the set of leaves of A .

Lemma 5.5. *For every dendroidally ordered set A the equality $d(A) = |A| - |L(A)|$ holds.*

5.2. The equivalence between the operadic approach and the dendroidal order approach. The constructions $T \mapsto [T]$ and $A \mapsto Tr(A)$ set up a correspondence between the trees depicted somewhat loosely in Section 2 and dendroidally ordered sets. We now have the categories Ω_c and Ω_π given above and the categories Ω^O and Ω_π^O given in [8] in terms of operads (denoted there by Ω and Ω_p).

Theorem 5.6. *There is an equivalence of categories $\Omega^O \cong \Omega_c$ and $\Omega_\pi^O \cong \Omega_\pi$.*

Proof. The constructions $A \mapsto Tr(A)$ and $T \mapsto [T]$ are easily seen to extend to functors establishing the desired equivalences. \square

Evidently, this equivalence establishes a translation mechanism from tree concepts to the language of dendroidally ordered sets. This is the tree formalism we propose. From this point onwards the term 'tree' is synonymous with 'dendroidally ordered set', and thus, conforming with our convention, comes in two flavours: commutative and non-commutative. Thus, 'tree' is meant to be replaced throughout by either 'commutative tree' or 'non-commutative tree'.

6. FACE-DEGENERACY FACTORIZATION

We give a characterization of the maximal subtrees of a given trees by means of pruning and contraction operations and prove a fundamental decomposition result for arrows in the dendroidal category.

6.1. Maximal subtrees. For the rest of this subsection fix a tree A of degree n and $B \subseteq A$ a subtree (i.e., B with the induced broad poset structure is dendroidally ordered) of degree $n - 1$ (such subtrees are called *maximal*). We also work under the extra assumption that B contains the root of A (necessarily as the root of B too). If that is not the case then the proofs below can be adapted to yield the same bottom line, but we omit the details.

First we notice that $L(A) \cap L(B) = L(A) \cap B$ always holds. Denote by k_1 the number of leaves of A that B misses, by k_2 the number of non-leaves of A that B misses, by t_1 the number of leaves in B that are also leaves in A , and by t_2 the number of leaves in B that are not leaves in A . By the degree formula in Lemma 5.5 we may write

$$d(A) = |A| - |L(A) \cap B| - |L(A) - B|$$

and

$$d(B) = |B| - |L(B) \cap L(A)| - |L(B) - L(A)|.$$

Subtraction yields $1 = k_2 + t_2$ and we analyze all possibilities. If $k_2 = 0$ then B only misses leaves of A , and there is precisely one leaf in B which is not a leaf in A . If, as sets, $A = B$ then B does not miss any edges of A and the only way to then create a new leaf is by omitting a vertex of the form $\epsilon \leq x$ for a unique x . Otherwise, $B \subset A$ and B misses at least one leaf $l \in L(A)$. We have that $l \in e^\uparrow$ for a unique edge e , which is not a leaf in A , and thus $e \in B$. We claim that B misses every child of e . Indeed, assume that e_1, \dots, e_k , with $k > 0$, are the children of e not missed by B . In B these edges are incomparable and are descendants of e . Thus, by Proposition 4.4, there is an element $u \in B^*$ such that $u \leq e$ and $e_i \in u$ for all $1 \leq i \leq k$. But then $u \leq e$ holds in A and thus $u \leq e^\uparrow$ which contradicts $l \in e^\uparrow$ being a leaf. Since B only misses leaves of A we conclude that every child of e is a leaf and so B misses the vertex $e \leq e^\uparrow$ which makes e into a new leaf, the only possible new one. Thus, the case $k_2 = 0$ implies that B is obtained either by turning a stump into a leaf or by pruning an outer cluster. If $k_2 = 1$ then B misses exactly one non-leaf e and no new leaves are present in B . But then $k_1 = 0$ since omitting a leaf of A must create a new leaf. Thus, B is obtained from A by omitting a single inner edge e .

We summarize these results as follows. Given a tree A of degree n there are three ways to produce a maximal subtree B . One is by omitting an inner edge $e \in A$, denoted by $B = A/e$. Another is by taking $B = A$ and omitting a stump $\epsilon \leq x$, and the third one is by pruning an outer cluster C , denoted by A/C . The

second of the three will also be considered a removal of an outer cluster. Above we established the following result.

Theorem 6.1. *Let A be a tree of degree n . If $B \subseteq A$ is a maximal subtree then $B = A/a$ for a unique inner edge $a \in A$ or $B = A/C$ for a unique outer cluster C (the meaning of 'or' should be taken in the exclusive sense).*

An inclusion $A/a \rightarrow A$ for an inner edge a is called an *inner face map*. An inclusion $A/C \rightarrow A$ for an outer cluster C is called an *outer face map*.

Example 6.2. If A is a dendroidally ordered set with root r and $b \in r^\uparrow$ then the inclusion $A_b \rightarrow A$ is a composition of outer face maps. To see that, notice that if $r^\uparrow = b$ then the vertex $r^\uparrow \leq r$ is an outer cluster and removing it gives A_b . Otherwise, there must be an outer cluster in A which is disjoint from A_b . Removing such outer clusters one at a time will eventually allow removing the root vertex and obtain A_b .

One more type of monotone function is the following one. Let $l = (a_1, a_2)$ be a unary vertex in A . The monotone function $\sigma_l : A \rightarrow A/a_2$ defined by

$$\sigma_l(x) = \begin{cases} x & x \neq a_2 \\ a_1 & x = a_2 \end{cases}$$

is called the *degeneracy map* associated with the unary vertex l .

Considering isomorphisms of trees we note that if $f : A \rightarrow B$ is an isomorphism then $f(r_A) = r_B$ and for every edge $a \in A$ the equality

$$f(a^\uparrow) = f(a)^\uparrow$$

holds. Obviously, the only isomorphisms in Ω_π are identities.

6.2. Fundamental decomposition of arrows in Ω . We now prove that every arrow in Ω decomposes as a composition of degeneracies, an isomorphism, and face maps. This result first appeared [9] without proof and more recently, with proof, as Lemma 2.3.2 in [8]. We also mention Lemma 1.3.17 in [7] that establishes essentially the same result using the polynomial functors formalism of trees. The following technical result is easily established.

Proposition 6.3. *If the map $\alpha : B \rightarrow B'$ of trees is an inner face (respectively outer face, degeneracy, isomorphism) then for any tree A , the map $A \circ \alpha : A \circ B \rightarrow A \circ B'$ is an inner face (respectively outer face, degeneracy, isomorphism) whenever the grafting is defined.*

Lemma 6.4. *Any arrow $f : A \rightarrow B$ in Ω decomposes as*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow \delta & & \uparrow \varphi \\ A' & \xrightarrow{\pi} & B' \end{array}$$

where $\delta : A \rightarrow A'$ is a composition of degeneracy maps, $\pi : A' \rightarrow B'$ is an isomorphism, and $\varphi : B' \rightarrow B$ is a composition of face maps.

Proof. The proof is by induction on $n = d(A) + d(B)$, noting that if at any point $d(A) = 0$ then the claim is trivial. The cases $n = 0$ and $n = 1$ are dealt with by inspection. Assume the assertion holds for $1 \leq n < m$ and assume $f : A \rightarrow B$ with

$|A| + |B| = m$. First assume that $f(r_A) = b \neq r_B$. In that case f factors through the inclusion $B_b \rightarrow B$, which by Example 6.2 is a composition of outer face maps. The induction hypothesis now furnishes the desired composition.

We now consider the case where $f(r_A) = r_B$ and $f(r_A^\uparrow) = r_B^\uparrow$. Let $r_A^\uparrow = a_1 \cdots a_k$ and $r_B^\uparrow = b_1 \cdots b_k$ with $f(a_i) = b_i$. In that case, by restricting f to A_{a_i} , one obtains the map $f_i : A_{a_i} \rightarrow B_{b_i}$. Let $A_{root} = \{r_A, a_1, \dots, a_k\}$ with the broad order induced by A and define B_{root} similarly. Let $f_{root} : A_{root} \rightarrow B_{root}$ be the restriction of f to A_{root} . The map f can be written as $f_{root} \circ (f_{a_1}, \dots, f_{a_k})$. The induction hypothesis then manufactures a decomposition of each f_i which can then be grafted together to produce the desired decomposition of f .

The third case is when $f(r_A) = r_B$ but $f(r_A^\uparrow) \neq r_B^\uparrow$. Notice that if $f(a) = r_B$ for some $a \in r_A^\uparrow$ then $r_A^\uparrow = a$ (otherwise \leq in B will not be finite) and thus (r_A, a) is a vertex. Let $\sigma : A \rightarrow A'$ be the degeneracy associated with it. Since $f(r_A) = f(a) = r_B$ it follows that f factors through σ as $f = f' \circ \sigma$. The induction hypothesis applied to f' together with the degeneracy σ produces the required decomposition of f . We may thus assume further that $f(a) \neq r_B$ for all $a \in r_A^\uparrow$ which, together with our assumption that $f(r_A^\uparrow) \neq r_B^\uparrow$, implies that the set $I = \{x \in B \mid r_B <_d x <_d f(a), a \in r_A^\uparrow\}$ is non-empty and consists entirely of inner edges. Let \hat{B} be the dendroidally ordered subset of B obtained by removing all of those inner elements. The inclusion $\hat{\phi} : \hat{B} \rightarrow B$ is then obviously a composition of (inner) face maps, and the map f factors as $f = \hat{\phi} \circ \hat{f}$. The induction hypothesis applied to \hat{f} together with $\hat{\phi}$ gives the desired decomposition of f and completes the proof. \square

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